2/11/26 Quiz!

Matrix Products and composition of linear transformations

Running Ex

$$\overset{B}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}}_{(4\times 2)} \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix}}_{(2\times 3)} = \begin{bmatrix} 3 & 4 & 1 \\ 11 & 16 & 9 \\ 5 & 6 & -1 \\ 6 & 10 & 10 \end{bmatrix}_{(4\times 3)} \leftarrow BA$$

what does this have to do with lin. transformations or with composition of functions?

Say $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ is def. by $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix} = A$ and

$T_2 : \mathbb{R}^2 \to \mathbb{R}^4$ is def by $\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} = B$

The composition $T_1 \circ T_2$ is undefined(!!)

since $T_2 : \mathbb{R}^2 \to \mathbb{R}^4$ means after you're done applying T_2 you're in $\mathbb{R}^4$ and T_1 can't touch you!

But $T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^4$ is well-defined.

$$(T_2 \circ T_1)(\vec{x}) = \underbrace{T_2(\overbrace{T_1(\underset{\uparrow \text{ in } \mathbb{R}^3}{\vec{x}})}^{\text{in } \mathbb{R}^2})}_{\text{in } \mathbb{R}^4}$$

How about matrices?

$T_1(\vec{x}) = A\vec{x}$

$T_2(\vec{y}) = B\vec{y}$

associativity

$$(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x})) = T_2(A\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$$

So the matrix for $T_2 \circ T_1$ is computed above as BA

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We just saw $T_2 \circ T_1$ is well-defined.

As a linear transf. $\mathbb{R}^3 \to \mathbb{R}^4$ it is defined by the matrix

$$\begin{bmatrix} 3 & 4 & 1 \\ 11 & 16 & 9 \\ 5 & 6 & -1 \\ 6 & 10 & 10 \end{bmatrix}$$

Continue Running Ex let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the standard vectors in $\mathbb{R}^3$. Then

$(T_2 \circ T_1)(\vec{e}_1) = \begin{bmatrix} 3 \\ 11 \\ 5 \\ 6 \end{bmatrix}$ $\quad (T_2 \circ T_1)(\vec{e}_2) = \begin{bmatrix} 4 \\ 16 \\ 6 \\ 10 \end{bmatrix}$

$(T_2 \circ T_1)(\vec{e}_3) = \begin{bmatrix} 1 \\ 9 \\ -1 \\ 10 \end{bmatrix}$

Matrix Algebra

Fact 1 – in general, commutativity may not even make sense for matrices.

e.g. $(2\times3)(3\times4) = (2\times4)$ but $(3\times4)(2\times3)$ is undefined.

- this isn't an issue for square matrices!
- but even for square matrices of the same size, matrix mult. is usually not commutative. (Th 2.3.3)

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 6 & 8 \end{bmatrix}$

Fact 2 However, it is associative! (You don't need them to be square as long as the products are defined.) (Th 2.3.6)

Ex $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} 2 & 1 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 8 \end{bmatrix}\right)$

(I leave it to you to check.)

Fact 3 Distributive Property (Th 2.3.7)

If A and B are $(n\times p)$ and C, D are $(p\times m)$ then

$$A(C+D) = AC + AD \quad \text{and} \quad (A+B)C = AC + BC$$

Fact 4 If I is the $n\times n$ Identity matrix, and A is $n\times n$,

then $AI = IA = A$. (Th 2.3.5)

Fact 5 scalar mult.

Assume A is $n\times p$

B is $p\times m$

k is a scalar.

Then $(kA)B = A(kB) = k(AB)$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $k = 2$

$$\Rightarrow \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix} = 2\left(\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 38 & 44 \\ 86 & 100 \end{bmatrix}$$

Reality check:

We've seen:

- linear transformations are linear (in precise sense)
- " " come from matrix mult. (incl. distributive property)

Let's see how those fit together - linearity is mostly from the distributive property

(i) $T(a\vec{v} + b\vec{w}) = aT(\vec{v}) + bT(\vec{w})$

$$A\left(a\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + b\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}\right) = aA\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + bA\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

(ii) $(T_2 \circ T_1)(\vec{v} + \vec{w}) = T_2(T_1(\vec{v}) + T_1(\vec{w})) = T_2(T_1(\vec{v})) + T_2(T_1(\vec{w}))$

$$(BA)\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}\right) = B\left(A\begin{bmatrix} \end{bmatrix} + A\begin{bmatrix} \end{bmatrix}\right) = BA\begin{bmatrix} \end{bmatrix} + BA\begin{bmatrix} \end{bmatrix}$$